Contractive metrics for scalar conservation laws

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Abstract

We consider nondecreasing entropy solutions to 1-d scalar conservation laws and show that the spatial derivatives of such solutions satisfy a contraction property with respect to the Wasserstein distance of any order. This result extends the L^1-contraction property shown by Kružkov.

Existence and uniqueness of solutions to scalar conservation laws in one space dimension have been established by Kružkov in the framework of entropy solutions (see [4] for instance), and among the properties satisfied by these solutions it is known that the L^1 norm between any two of them is a non-increasing function of time.

In this work we shall focus on a class of entropy solutions such that a certain distance between the space derivatives of any two such solutions is also nonincreasing in time. On this class of solutions this result extends the L^1 norm contraction property.

More precisely we consider as initial data nondecreasing functions on $\mathbb{R}$ with limits 0 and 1 at $-\infty$ and $+\infty$ respectively. These properties are preserved by the conservation law, and corresponding solutions have been shown in [2] to arise in some models of pressureless gases, obtained as a continuous limit of systems of sticky particles. Noticing that the distributional space derivative of these functions are probability measures, we may consider the Wasserstein distance between the space derivatives of any two such solutions, and we shall prove in this paper that this distance is a nonincreasing function of time, constant in the case of classical solutions.

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1   Introduction to the results

Given a locally Lipschitz real-valued function f on $\mathbb{R}$, called a flux, we consider the scalar conservation law

$$(1) \qquad \begin{cases} u_t + f(u)_x = 0\,, & t > 0,\ x \in \mathbb{R}, \\ u(0,.) = u^0, \end{cases}$$

with unknown $u = u(t,x) \in \mathbb{R}$ and initial datum $u^0 \in L^\infty(\mathbb{R})$, and where the subscripts stand for derivation.

We shall consider solutions that are called *entropy solutions* (see [5] for instance) and are defined as follows: a function $u = u(t,x) \in L^\infty([0,+\infty[\times\mathbb{R})$ is said to be an entropy solution of (1) on $[0,+\infty[\times\mathbb{R}$ if the entropy inequality

$$(2) \qquad\qquad E(u)_t + F(u)_x \leq 0$$

holds in the sense of distributions for all convex Lipschitz function E on $\mathbb{R}$, and with associated flux F defined by

$$(3) \qquad\qquad F(u) = \int_0^u f'(v)\,E'(v)\,dv.$$

This means that

$$\int_0^{+\infty}\!\!\int_{\mathbb{R}} \big(E(u)\,\varphi_t + F(u)\,\varphi_x\big)\,dt\,dx \;+\; \int_{\mathbb{R}} E(u^0(x))\,\varphi(0,x)\,dx \;\geq 0$$

for all nonnegative φ in the space $\mathcal{C}_c^\infty([0,+\infty[\times\mathbb{R})$ of $\mathcal{C}^\infty$ functions on $[0,+\infty[\times\mathbb{R}$ with compact support.

We shall also consider *classical solutions*, that is, functions $u = u(t,x)$ in $\mathcal{C}^1(]0,+\infty[\times\mathbb{R})\cap \mathcal{C}([0,+\infty[\times\mathbb{R})$ satisfying (1) pointwise.

In particular any classical solution to (1) satisfies (2), i.e. is an entropy solution, and conversely any entropy solution satisfies (1) in the distribution sense.

For entropy solutions, the following result is due to Kružkov (see [4]):

Theorem 1.1 *For every $u^0 \in L^\infty(\mathbb{R})$, there exists a unique entropy solution u to (1) in $L^\infty([0,+\infty[\times\mathbb{R}) \cap \mathcal{C}([0,+\infty[, L^1_{loc}(\mathbb{R})).$*

Moreover for classical solutions, we have (see [5] for instance):

Theorem 1.2 *Given a $\mathcal{C}^2$ flux f and a $\mathcal{C}^1$ bounded initial datum u^0 such that $f' \circ u^0$ is nondecreasing on $\mathbb{R}$, the unique entropy solution u to (1) is a classical solution.*

In this work we shall consider initial data in the subset $\mathcal{U}$ of $L^\infty(\mathbb{R})$ defined by

Definition 1.3 *A function $v : \mathbb{R} \to \mathbb{R}$ belongs to $\mathcal{U}$ if it is nondecreasing, right-continuous, and has limits 0 and 1 at $-\infty$ and $+\infty$ respectively.*

The following proposition expresses that this set is preserved by the conservation law (1):

Proposition 1.4 *Given an initial datum $u^0 \in \mathcal{U}$, the entropy solution u given by Theorem 1.1 is such that $u(t, .)$ belongs to $\mathcal{U}$ for all $t \geq 0$.*

More precisely, given any $t \geq 0$, the $L^\infty(\mathbb{R})$ function $u(t, .)$ is *a.e.* equal to an element of the set $\mathcal{U}$, which on the other hand is characterized by

Proposition 1.5 *The distributional derivative v_x of any $v \in \mathcal{U}$ is a Borel probability measure on $\mathbb{R}$, and for any $x \in \mathbb{R}$,*

$$v(x) = v_x(]-\infty, x]).$$

Conversely, if μ is a probability measure on $\mathbb{R}$, then v defined on $\mathbb{R}$ as

$$v(x) = \mu(]-\infty, x])$$

belongs to $\mathcal{U}$, and $v_x = \mu$.

Consequently the map $v \mapsto v_x$ is one-to-one from $\mathcal{U}$ onto the set $\mathcal{P}$ of probability measures on $\mathbb{R}$ (and $\mathcal{U}$ can be seen as the set of repartition functions of real-valued random variables).

Propositions 1.4 and 1.5 allow us to characterize at any time the distance between two solutions (with initial datum in $\mathcal{U}$) in terms of their space derivatives, in particular by means of the Wasserstein distances: given any real number $p \geq 1$, the Wasserstein distance of order p is defined on the set of probability measures on $\mathbb{R}$ by

$$W_p(\mu, \tilde{\mu}) = \inf_\pi \left(\int_{\mathbb{R}^2} |x - y|^p \, d\pi(x, y) \right)^{1/p}$$

where π runs over the set of probability measures on $\mathbb{R}^2$ with marginals μ and $\tilde{\mu}$; these distances are considered here in a broad sense with possibly infinite values.

This paper aims at proving that the Wasserstein distances between the space derivatives of any two such entropy solutions is a nonincreasing function of time:

Theorem 1.6 *Given a locally Lipschitz real-valued function f on $\mathbb{R}$ and two initial data u^0 and $\tilde{u}^0$ in $\mathcal{U}$, let u and $\tilde{u}$ be the associated entropy solutions to (1). Then, for any $t \geq 0$ and $p \geq 1$, we have (with possibly infinite values)*

$$W_p(u_x(t, .), \tilde{u}_x(t, .)) \leq W_p(u^0_x, \tilde{u}^0_x).$$

We shall see in Section 2 that for $p = 1$ the distance W_1 satisfies

$$W_1(v_x, \tilde{v}_x) = \|v - \tilde{v}\|_{L^1(\mathbb{R})}$$

for all $v, \tilde{v} \in \mathcal{U}$. Hence Theorem 1.6 reads in the case $p = 1$:

$$\|u(t, .) - \tilde{u}(t, .)\|_{L^1(\mathbb{R})} \leq \|u^0 - \tilde{u}^0\|_{L^1(\mathbb{R})}.$$

Thus, for initial profiles in $\mathcal{U}$, we recover the L^1-contraction property given by Kružkov.

In the case of classical solutions, the result of Theorem 1.6 is improved, since the Wasserstein distance between two solutions is conserved:

Theorem 1.7 *Given a $\mathcal{C}^1$ real-valued function f on $\mathbb{R}$, let u^0 and $\tilde{u}^0$ in $\mathcal{U}$ be two initial data such that the associated entropy solutions u and $\tilde{u}$ to (1) are classical solutions, increasing in x for all $t \geq 0$. Then for any $t \geq 0$ and $p \geq 1$ we have (with possibly infinite values)*

$$W_p(u_x(t, .), \tilde{u}_x(t, .)) = W_p(u^0_x, \tilde{u}^0_x).$$

From these general results can be induced some corollaries in the case of initial data in the subsets $\mathcal{U}_p$ of $\mathcal{U}$ defined as:

Definition 1.8 *Let $p \geq 1$. A function v in $\mathcal{U}$ belongs to $\mathcal{U}_p$ if its distributional derivative v_x has finite moment of order p, that is, if $\int_{\mathbb{R}} |x|^p \, dv_x(x)$ is finite.*

As in Proposition 1.5 the map $v \mapsto v_x$ is one-to-one from $\mathcal{U}_p$ onto the set $\mathcal{P}_p$ of probability measures on $\mathbb{R}$ with finite moment of order p. But we shall note in Section 2 that the map W_p on $\mathcal{P}_p \times \mathcal{P}_p$ defines a distance on $\mathcal{P}_p$. Then the real-valued map d_p defined on $\mathcal{U}_p \times \mathcal{U}_p$ by

$$d_p(v, \tilde{v}) = W_p(v_x, \tilde{v}_x)$$

induces a distance on $\mathcal{U}_p$, and for the associated topology we have

Corollary 1.9 *Given a locally Lipschitz function f on $\mathbb{R}$, $p \geq 1$ and $u^0 \in \mathcal{U}_p$, the entropy solution u to (1) belongs to $\mathcal{C}([0, +\infty[, \mathcal{U}_p)$.*

In particular for $p = 1$

$$d_1(v, \tilde{v}) = W_1(v_x, \tilde{v}_x) = \|v - \tilde{v}\|_{L^1(\mathbb{R})},$$

and the previous result can be precised by

Corollary 1.10 *Given a locally Lipschitz function f on $\mathbb{R}$ and $u^0 \in \mathcal{U}_1$, the entropy solution u to (1) is such that*

$$\|u(t, .) - u(s, .)\|_{L^1(\mathbb{R})} \leq |t - s| \, \|f'\|_{L^\infty([0,1])}.$$

This known result holds under weaker assumptions (for u^0 with bounded variation, see [5]), but in our case it will be recovered in a straightforward way.

Finally Theorem 1.7 can be precised in the $\mathcal{U}_p$ framework in the following way:

Corollary 1.11 *Given a $\mathcal{C}^2$ convex flux f and two $\mathcal{C}^1$ increasing initial data u^0 and $\tilde{u}^0$ in $\mathcal{U}_p$ for some $p \geq 1$, the following three properties hold:*
1. *the associated entropy solutions u and $\tilde{u}$ are classical solutions;*
2. *$u(t,.)$ and $\tilde{u}(t,.)$ belong to $\mathcal{U}_p$ and are increasing for all $t \geq 0$;*
3. *for all $t \geq 0$, we have (with finite values)*

$$W_p(u_x(t,.), \tilde{u}_x(t,.)) = W_p(u_x^0, \tilde{u}_x^0).$$

The paper is organized as follows. The definition and some properties of Wasserstein distances are discussed in greater detail in Section 2. In Section 3 we consider the case of classical solutions, proving Theorem 1.7 and Corollary 1.11. Then the general case of entropy solutions is studied in Sections 4 and 5: more precisely in Section 4 we introduce a time-discretized scheme, show the W_p contraction property for this discretized evolution and prove the convergence of the corresponding approximate solution toward the entropy solution; Theorem 1.6 and its corollaries follow from this in Section 5. In Section 6 we shall finally see how such results extend to viscous conservation laws.

2   Wasserstein distances

In this section p is a real number with $p \geq 1$, $\mathcal{P}$ (resp. $\mathcal{P}_p$) stands for the set of probability measures on $\mathbb{R}$ (resp. with finite moment of order p) and dx for the Lebesgue measure on $\mathbb{R}$.

The Wasserstein distance of order p, valued in $\mathbb{R} \cup \{+\infty\}$, is defined on $\mathcal{P} \times \mathcal{P}$ by

$$(4) \qquad W_p(\mu, \tilde{\mu}) = \inf_{\pi} \left(\int_{\mathbb{R}^2} |x - y|^p \, d\pi(x, y) \right)^{1/p}$$

where π runs over the set of probability measures on $\mathbb{R}^2$ with marginals μ and $\tilde{\mu}$. It is equivalently defined by

$$(5) \qquad W_p(\mu, \tilde{\mu}) = \inf_{X_\mu, X_{\tilde{\mu}}} \left(\int_0^1 |X_\mu(w) - X_{\tilde{\mu}}(w)|^p \, dw \right)^{1/p}$$

where the infimum is taken over all random variables X_μ and $X_{\tilde{\mu}}$ on the probability space $(]0, 1[, dw)$ with respective laws μ and $\tilde{\mu}$. It takes finite values on $\mathcal{P}_p \times \mathcal{P}_p$ and indeed defines a distance on $\mathcal{P}_p$.

For complete references about the Wasserstein distances and related topics the reader can refer to [6]. We only mention that both infima in (4) and (5) are achieved, and for the second definition we shall precise some random variables that achieve the infimum. For this purpose we introduce the notion of generalized inverse:

Definition 2.1 *Let v belong to $\mathcal{U}$. Then its generalized inverse is the function v^{-1} defined on $]0,1[$ by*

$$v^{-1}(w) = \inf\{x \in \mathbb{R}; v(x) > w\}.$$

Then v^{-1} is a nondecreasing random variable on $(]0,1[,dw)$ by definition, with law v_x since

$$
\begin{aligned}
\int_0^1 f(v^{-1}(w)) \, dw &= \int_0^1 \left(\int_{\mathbb{R}} f'(s) \mathbf{1}_{\{s \leq v^{-1}(w)\}} \, ds \right) dw \\
&= \int_{\mathbb{R}} \left(\int_0^1 f'(s) \mathbf{1}_{\{v(s) \leq w\}} \, dw \right) ds \\
&= \int_{\mathbb{R}} f'(s)(1 - v(s)) \, ds \\
&= \int_{\mathbb{R}} f(s) \, dv_x(s)
\end{aligned}
$$

for all f in $\mathcal{C}_c^1(\mathbb{R})$. In particular its repartition function is v.

Moreover this generalized inverse achieves the infimum in (5):

Proposition 2.2 *Let v and $\tilde{v}$ in $\mathcal{U}$. Then we have (with possibly infinite values)*

$$W_p(v_x, \tilde{v}_x) = \left(\int_0^1 |v^{-1}(w) - \tilde{v}^{-1}(w)|^p \, dw \right)^{1/p}$$

for all $p \geq 1$. In particular for $p = 1$ we also have

$$W_1(v_x, \tilde{v}_x) = \|v - \tilde{v}\|_{L^1(\mathbb{R})}.$$

Proof. The general result is proved in [6]. The result specific to the case $p = 1$ follows by introducing, for a given $v \in \mathcal{U}$, the map defined on $\mathbb{R} \times]0,1[$ by

$$jv(x,w) = \begin{cases} 1 & \text{if } v(x) > w \\ 0 & \text{if } v(x) \leq w, \end{cases}$$

for which we have

$$|v^{-1} - \tilde{v}^{-1}|(w) = \int_{\mathbb{R}} |jv - j\tilde{v}|(x,w) \, dx$$

for almost every $w \in]0,1[$, and

$$\int_0^1 |jv - j\tilde{v}|(x,w) \, dw = |v - \tilde{v}|(x)$$

for almost every $x \in \mathbb{R}$. Integrating the first equality on w in $]0,1[$ and the second one on x in $\mathbb{R}$, we deduce

$$\int_0^1 |v^{-1} - \tilde{v}^{-1}|(w) \, dw = \int_{\mathbb{R}} |v - \tilde{v}|(x) \, dx.$$

$\square$

Given $v \in \mathcal{U}$, its generalized inverse v^{-1} is actually the a.e. unique nondecreasing random variable on $(]0,1[, dw)$ with law v_x. Given any other random variable X on $(]0,1[, dw)$ with law v_x, v^{-1} is called the (a.e. unique) nondecreasing rearrangement of X (see [6]).

We conclude this section recalling a result relative to the convergence of probability measures. A sequence (μ_n) of probability measures on $\mathbb{R}$ is said to converge weakly toward a probability measure μ if, as n goes to $+\infty$, $\int_{\mathbb{R}} \varphi \, d\mu_n$ tends to $\int_{\mathbb{R}} \varphi \, d\mu$ for all bounded continuous real-valued functions φ on $\mathbb{R}$ (or equivalently for all $\mathcal{C}^\infty$ functions φ with compact support, that is, if μ_n converges to μ in the distribution sense). Given $p \geq 1$ this convergence is metrized on $\mathcal{P}_p$ by the distance W_p as shown by the following proposition (see [6]):

Proposition 2.3 *Let $p \geq 1$, (μ_n) a sequence of probability measures in $\mathcal{P}_p$ and $\mu \in \mathcal{P}$. Then the following statements are equivalent:*
   i) $(W_p(\mu_n, \mu))$ converges to 0;
   ii) (μ_n) converges weakly to μ and $\sup_n \int_{|x| \geq R} |x|^p \, d\mu_n(x)$ tends to 0 as R goes to infinity.

In this proposition we do not a priori assume that μ belongs to $\mathcal{P}_p$, but it can be noted that this property is actually induced by any of both hypotheses $i)$ and $ii)$.

For measures in $\mathcal{P}$ we have the weaker result:

Proposition 2.4 *Let $p \geq 1$, (μ_n) and (ν_n) two sequences in $\mathcal{P}$ converging weakly to μ and ν in $\mathcal{P}$ respectively. Then (with possibly infinite values)*

$$W_p(\mu, \nu) \leq \liminf_{n \to +\infty} W_p(\mu_n, \nu_n).$$

3   The case of classical solutions: Theorem 1.7 and corollary

3.1   Proof of Theorem 1.7

We consider two classical solutions u and $\tilde{u}$ to (1) such that $u(t, .)$ and $\tilde{u}(t, .)$ belong to $\mathcal{U}$ and are increasing for all $t \geq 0$, and we shall prove that

$$W_p(u_x(t, .), \tilde{u}_x(t, .)) = W_p(u_x^0, \tilde{u}_x^0)$$

as a consequence of Proposition 2.2.

The map u^0 is increasing from 0 to 1, so has a (true) inverse $X(0,.)$ defined on $]0, 1[$ by

$$u^0(X(0, w)) = w.$$

Then, given $w \in]0, 1[$, we consider a characteristic curve $t \mapsto X(t, w)$ solution of

$$(6) \qquad\qquad X_t(t, w) = f'\big(u(t, X(t, w))\big)$$

for $t \geq 0$, and taking value $X(0, w)$ at $t = 0$. Since f is $\mathcal{C}^1$ and u is bounded there exists a (non necessarily unique) solution $X(., w)$ to (6) by Peano Theorem (see [3] for instance); moreover by a classical computation from (1) it is known to satisfy

$$(7) \qquad\qquad u(t, X(t, w)) = w$$

for all $t \geq 0$, from which it follows that

$$X_t(t, w) \left(= f'(u(t, X(t, w)))\right) = f'(w)$$

and hence

$$(8) \qquad\qquad X(t, w) = X(0, w) + t f'(w).$$

In particular there exists a unique solution $X(., w)$ to (6). Now given $t \geq 0$, $X(t, .)$ is the (true) inverse of the increasing function $u(t, .)$ (by (7)), and Proposition 2.2 writes

$$W_p(u_x(t, .), \tilde{u}_x(t, .)) = \left(\int_0^1 |X(t, w) - \tilde{X}(t, w)|^p \, dw \right)^{1/p}.$$

But from (8) we obtain

$$(9) \qquad\qquad X(t, w) - \tilde{X}(t, w) = X(0, w) - \tilde{X}(0, w).$$

This result ensures in particular that $W_p(u_x(t, .), \tilde{u}_x(t, .))$ remains constant in time, may its initial value be finite or not; note however that (9) is actually much stronger that Theorem 1.7.

3.2   Proof of Corollary 1.11

We assume that f is a $\mathcal{C}^2$ convex function on $\mathbb{R}$, and u^0 is a $\mathcal{C}^1$ increasing initial profile in $\mathcal{U}_p$.

First of all we note that the associated entropy solutions u is a classical solution in view of Theorem 1.2: this result is proved in [5] for instance, and its proof also ensures that $u(t, .)$ is increasing for all $t \geq 0$.

Then we check that the moment property is preserved by the conservation law, that is, that $u(t, .)$ also belongs to $\mathcal{U}_p$ for any $t \geq 0$. Indeed, given $t \geq 0$, we have by the change of variable $w = [u(t, .)](x)$:

$$
\begin{aligned}
\int_{\mathbb{R}} |x|^p \, u_x(t, x) \, dx & = \int_0^1 |X(t, w)|^p \, dw \\
& = \int_0^1 |X(0, w) + t f'(w)|^p \, dw \\
& \leq 2^{p-1} \left[\int_0^1 |X(0, w)|^p \, dw + t^p \|f'\|_{L^\infty(]0,1[)}^p \right]
\end{aligned}
$$

which is finite since

$$
\int_0^1 |X(0, w)|^p \, dw = \int_{\mathbb{R}} |x|^p \, u_x^0(x) \, dx
$$

is finite by assumption. This ends the proof of Corollary 1.11.   □

4   Time discretization of the conservation law

In the previous section we have seen that the classical solutions are obtained through the method of characteristics, that we now summarize in our case: given an initial profile u^0 in $\mathcal{U}$ such that the corresponding solution u is $\mathcal{C}^1$ and increasing in x for all $t \geq 0$, let $X(0, .)$ be its inverse, defined by

$$
u^0(X(0, w)) = w
$$

for all $w \in]0, 1[$. Let then $X(0, w)$ evolve into

$$
(10) \qquad\qquad X(t, w) = X(0, w) + t f'(w)
$$

for all $t \geq 0$ and $w \in]0, 1[$ (see (8)). The solution $u(t, .)$ is then the inverse of the increasing map $X(t, .)$, that is, is the unique solution of

$$
X(t, u(t, x)) = x.
$$

In the general case, defining $X(0, .)$ in some similar way, there is no hope for the function $X(t, .)$ defined by (10) to be increasing for $t > 0$; inverting it would thus lead to a multivalued function, and no more to the entropy solution of the conservation law, as in the particular case discussed above.

However, averaging (or "collapsing") this multivalued function into a single-valued function, Y. Brenier showed in [1] how to build an approximate solution to the conservation law.

We now precisely describe this so-called Transport-Collapse method in our case.

4.1   Definition and W_p contraction property of the discretized solution

Let $u^0 \in \mathcal{U}$ be some fixed initial profile, with generalized inverse $X(0, .)$ given as in Definition 2.1 by

$$X(0, w) \ = \ \inf\{x \in \mathbb{R}; u^0(x) > w\}$$

for all $w \in]0, 1[$. $X(0, .)$ can be seen as a random variable on the probability space $]0, 1[$ equipped with the Lebesgue measure dw; its law is u_x^0, as pointed out after Definition 2.1.

We let then $X(0, .)$ evolve according to the method of characteristics, denoting

$$X(h, w) \ = \ X(0, w) \ + \ hf'(w)$$

for all $h \geq 0$ and almost every $w \in]0, 1[$. Again, given $h \geq 0$, $X(h, .)$ can be seen as a random variable on $]0, 1[$; let then $T_h u^0$ be its repartition function, that is, the function belonging to $\mathcal{U}$ and defined at any $x \in \mathbb{R}$ as the Lebesgue measure of the set $\{w \in]0, 1[; X(h, w) \leq x\}$. It is given by

$$T_h u^0(x) \ = \ \int_0^1 \mathbf{1}_{\{X(h,w) \leq x\}}(w) \, dw.$$

We summarize this construction in the following definition:

Definition 4.1 *Let $v \in \mathcal{U}$ with generalized inverse $X(0, .)$ defined on $]0, 1[$ by*

$$X(0, w) \ = \ \inf\{x \in \mathbb{R}; v(x) > w\}.$$

Then, given $h \geq 0$, and letting

$$X(h, w) \ = \ X(0, w) \ + \ hf'(w)$$

for almost every $w \in]0, 1[$, we define the $\mathcal{U}$ function $T_h v$ on $\mathbb{R}$ by

$$T_h v(x) \ = \ \int_0^1 \mathbf{1}_{\{X(h,w) \leq x\}}(w) \, dw.$$

In the case of Section 3 (see (8)), it turns out that $X(h, .)$ is the (true) inverse of $T_h u^0$, and $(h, x) \mapsto T_h u^0(x)$ is exactly the entropy solution to equation (1) with initial datum u^0 in $\mathcal{U}$. This does not hold anymore in the general case, but will allow us to build an approximate solution $S_h u^0$ by iterating the operator T_h. Let us first give two important properties of T_h:

Proposition 4.2 *Let $h \geq 0$, T_h defined as above and $p \geq 1$. Then*
i) $T_h v$ belongs to $\mathcal{U}_p$ if so does v.
ii) For any v and $\tilde{v}$ in $\mathcal{U}$ we have (with possibly infinite values unless v and $\tilde{v} \in \mathcal{U}_p$)

$$W_p([T_h v]_x, [T_h \tilde{v}]_x) \ \leq \ W_p(v_x, \tilde{v}_x).$$

Proof. It is really similar to what has been done in Section 3 as for Corollary 1.11.
i) $T_h v$ belongs to $\mathcal{U}$ as a repartition function of a random variable, and we have

$$
\begin{aligned}
\int_{\mathbb{R}} |x|^p \, d[T_h v]_x(x) &= \int_0^1 |X(h,w)|^p \, dw \\
&= \int_0^1 |X(0,w) + hf'(w)|^p \, dw \\
&\le 2^{p-1} \int_0^1 |X(0,w)|^p + |hf'(w)|^p \, dw \\
&\le 2^{p-1} \left[\int_{\mathbb{R}} |x|^p \, dv_x(x) + h^p \|f'\|_{L^\infty(]0,1[)}^p \right],
\end{aligned}
$$

which ensures that $[T_h v]_x$ has finite moment of order p if so does v_x.

ii) On one hand the generalized inverses $X(0,.)$ and $\tilde{X}(0,.)$ of v and $\tilde{v}$ respectively satisfy

$$
(11) \qquad W_p(v_x, \tilde{v}_x) = \left(\int_0^1 |X(0,w) - \tilde{X}(0,w)|^p \, dw \right)^{1/p}
$$

by Proposition 2.2 (with finite values if both v and $\tilde{v}$ belong to $\mathcal{U}_p$, and possibly infinite otherwise). On the other hand $X(h,.)$ and $\tilde{X}(h,.)$ have respective law $[T_h v]_x$ and $[T_h \tilde{v}]_x$, so

$$
(12) \qquad W_p([T_h u^0]_x, [T_h \tilde{u}^0]_x) \le \left(\int_0^1 |X(h,w) - \tilde{X}(h,w)|^p \, dw \right)^{1/p}
$$

by definition of the Wasserstein distance. But

$$
X(h,w) - \tilde{X}(h,w) = X(0,w) - \tilde{X}(0,w)
$$

for almost every $w \in \,]0,1[$ by definition, which concludes the argument by (11) and (12).

Note again that (12) holds only as an inequality since $X(h,.)$ and $\tilde{X}(h,.)$ are not necessarily nondecreasing, which was the case in the example discussed in Section 3. $\square$

We now use the operator T_h defined above to build an approximate solution $S_h u^0$ to the conservation law (1):

Definition 4.3 *Let h be some positive number and $v \in \mathcal{U}$. For any $t \ge 0$ decomposed as $t = (N+s)h$ with $N \in \mathbb{N}$ and $0 \le s < 1$, we let*

$$
S_h v(t,.) = (1-s) \, T_h^N v(.) + s \, T_h^{N+1} v(.)
$$

where $T_h^0 v = v$ and $T_h^{N+1} v = T_h(T_h^N v)$.

These iterations make sense because $T_h v \in \mathcal{U}$ if $v \in \mathcal{U}$, $S_h v(t, .) \in \mathcal{U}$ (resp. $\mathcal{U}_p$) for any $h, t \geq 0$ and $v \in \mathcal{U}$ (resp. $\mathcal{U}_p$).

We now prove two contractions properties on these approximate solutions. We first have the $L^1(\mathbb{R})$ contraction property:

Proposition 4.4 *Let h be some fixed positive number and S_h defined as above. Then, for any $v \in \mathcal{U}$ and $s, t \geq 0$ we have*

$$\|S_h v(t, .) - S_h v(s, .)\|_{L^1(\mathbb{R})} \leq |t - s| \, \|f'\|_{L^\infty([0,1])}.$$

Proof. As in [1] we first observe that $\|T_h V - V\|_{L^1(\mathbb{R})} \leq h\|f'\|_{L^\infty(]0,1[)}$ for any $V \in \mathcal{U}$, then let $t = (M + \mu)h$ and $s = (N + \nu)h$ with $M, N \in \mathbb{N}$ and $0 \leq \mu, \nu < 1$, and, for instance assuming that $M > N$, prove the proposition by applying this first bound to $V = S_h v(kh, .) = T_h^k v$ for $k = N + 1, \ldots, M - 1$. $\square$

Then we have the W_p contraction property:

Proposition 4.5 *Let h be some fixed positive number and S_h defined as above. Then, given v and $\tilde{v}$ in $\mathcal{U}$, we have for any $t \geq 0$:*

$$W_p([S_h v]_x(t, .), [S_h \tilde{v}]_x(t, .)) \leq W_p(v_x, \tilde{v}_x).$$

Proof. It follows from Proposition 4.2 (about T_h) and to the convexity of the W_p distance to the power p, in the sense that

$$W_p^p(\alpha\mu_1 + (1 - \alpha)\mu_2, \alpha\nu_1 + (1 - \alpha)\nu_2) \leq \alpha W_p^p(\mu_1, \nu_1) + (1 - \alpha)W_p^p(\mu_2, \nu_2)$$

for all real number $\alpha \in [0, 1]$ and probability measures μ_1, μ_2, ν_1 and ν_2 (see [6] for instance). $\square$

We shall now recall the convergence of the scheme toward the entropy solution of the conservation law.

4.2 Convergence of the scheme in the $L^1_{loc}(\mathbb{R})$ sense

In this section we consider the space $\mathcal{C}([0, +\infty[, L^1_{loc}(\mathbb{R}))$ equipped with the topology defined by the semi-norms

$$q_{nm}(f) = \sup_{t \in [0,n]} \int_{-m}^{m} |f(x)| \, dx$$

for any integers n and m and $f \in \mathcal{C}([0, +\infty[, L^1_{loc}(\mathbb{R}))$. Then we have

Proposition 4.6 *Let $u^0 \in \mathcal{U}$. Then, as h goes to 0, the function $S_h u^0$ converges in $\mathcal{C}([0, +\infty[, L^1_{loc}(\mathbb{R}))$ to the entropy solution of (1) with initial datum u^0.*

We briefly give the steps of the proof, which follows the one of Brenier in [1], adapted to functions of $\mathcal{U}$ instead of $L^1(\mathbb{R})$.

We first prove that the family $(S_h u^0)_h$ is relatively compact in $\mathcal{C}([0, +\infty[, L^1_{loc}(\mathbb{R}))$ by means of Proposition 4.4, and Helly and Ascoli-Arzela Theorems. Then we check that the limit of any sequence of $(S_h u^0)_h$ converging in $\mathcal{C}([0, +\infty[, L^1_{loc}(\mathbb{R}))$ is an entropy solution to the conservation law (1) with initial datum u^0. By the uniqueness of this solution ensured by Theorem 1.1, this concludes the proof of Proposition 4.6.

4.3   Convergence of the scheme in W_p distance sense

We first prove a uniform equiintegrability result on the approximate solutions:

Proposition 4.7 *Let S_h be defined as above, $v \in \mathcal{U}_p$ and $T \geq 0$. Then*

$$\sup_{0 \leq h \leq T} \sup_{0 \leq t \leq T} \int_{|x| \geq R} |x|^p d[S_h v]_x(t, x)$$

tends to 0 as R goes to infinity.

Proof. We again denote $M = \|f'\|_{L^\infty(]0,1[)}$, and first consider T_h itself, writing

$$
\begin{aligned}
\int_{|x| \geq R} |x|^p d[T_h v]_x(x) &= \int_0^1 |v^{-1}(w) + hf'(w)|^p \mathbf{1}_{\{|v^{-1}(w)+hf'(w)| \geq R\}}\, dw \\
&\leq \int_{\mathbb{R}} (|x| + hM)^p \mathbf{1}_{\{|x|+hM \geq R\}}\, dv_x(x) \\
&\leq \left(1 + \frac{hM}{R - hM}\right)^p \int_{|x| \geq R - hM} |x|^p\, dv_x(x)
\end{aligned}
$$

for $R > hM$. From this computation we deduce by iteration

$$\int_{|x| \geq R} |x|^p d[T_h^N v]_x(x) \leq \prod_{j=1}^N \left(1 + \frac{hM}{R - jhM}\right)^p \int_{|x| \geq R - NhM} |x|^p\, dv_x(x)$$

for $R > NhM$, with

$$\prod_{j=1}^N \left(1 + \frac{hM}{R - jhM}\right) \leq \left(1 + \frac{hM}{R - NhM}\right)^N \leq \exp\left(\frac{NhM}{R - NhM}\right).$$

Thus

$$\int_{|x| \geq R} |x|^p d[S_h v]_x(Nh, x) \leq \exp\left(\frac{pTM}{R - TM}\right) \int_{|x| \geq R - TM} |x|^p\, dv_x(x)$$

for any N and h such that $Nh \leq T$.

From this we get for instance

$$\int_{|x|\geq R} |x|^p d[S_h v]_x(t,x) \leq \exp\left(\frac{2pTM}{R-2TM}\right) \int_{|x|\geq R-2TM} |x|^p\, dv_x(x)$$

for any t and h smaller than T. This concludes the argument since the last integral tends to 0 as R goes to infinity. $\qquad\square$

From this we deduce the convergence of the scheme in W_p distance sense:

Proposition 4.8 *Let $u^0 \in \mathcal{U}_p$ and u be the entropy solution to (1) with initial datum u^0. Then, for any $t \geq 0$, $W_p([S_h u^0]_x(t,.), u_x(t,.))$ converges to 0 as h goes to 0.*

Proof. Given $t \geq 0$, $S_h u^0(t,.)$ converges to $u(t,.)$ in $L^1_{loc}(\mathbb{R})$ as h goes to 0 (by Proposition 4.6), so $[S_h u]_x(t,.)$ converges to the probability measure $u_x(t,.)$, first in the distribution sense, then in the weak sense of probability measures, and finally in W_p distance by Propositions 4.7 and 2.3.

Note in particular that $u_x(t,.)$ has finite moment of order p for any $t \geq 0$, that is, $u(t,.)$ belongs to $\mathcal{U}_p$. $\qquad\square$

5   The general case of entropy solutions: Theorem 1.6 and corollaries

5.1   Proof of Theorem 1.6

We let $p \geq 1$ and consider two initial data u^0 and $\tilde{u}^0$ in $\mathcal{U}$ with associated entropy solutions u and $\tilde{u}$.

Given $t \geq 0$, Proposition 4.6 yields again the convergence of $[S_h u^0]_x(t,.)$ to $u_x(t,.)$ in the weak sense of probability measures. Since this holds also for $\tilde{u}^0$, we obtain

$$W_p(u_x(t,.), \tilde{u}_x(t,.)) \leq \liminf_{h\to 0} W_p([S_h u^0]_x(t,.), [S_h \tilde{u}^0]_x(t,.))$$

by Proposition 2.4. But, for each h,

$$W_p([S_h u^0]_x(t,.), [S_h \tilde{u}^0]_x(t,.)) \leq W_p(u_x^0, \tilde{u}_x^0)$$

by Proposition 4.5, so finally

$$W_p(u_x(t,.), \tilde{u}_x(t,.)) \leq W_p(u_x^0, \tilde{u}_x^0).$$

This concludes the argument.

5.2   Proof of Corollary 1.9

We recall that in the introduction we have defined a distance on each $\mathcal{U}_p$ by letting

$$d_p(u, \tilde{u}) \; = \; W_p(u_x, \tilde{u}_x),$$

and we now prove that, given $p \geq 1$ and $u^0 \in \mathcal{U}_p$, the entropy solution u to the conservation law (1) belongs to $\mathcal{C}([0, +\infty[, \mathcal{U}_p)$.

We first note, in view of the proof of Proposition 4.8, that $u(t, .)$ indeed belongs to $\mathcal{U}_p$ for all $t \geq 0$.

Then, given $s \geq 0$, we need to prove that $d_p(u(t, .), u(s, .))\,(= W_p(u_x(t, .), u_x(s, .)))$ tends to 0 as t goes to s. Indeed, on one hand $u(t, .)$ tends to $u(s, .)$ in $L^1_{loc}(\mathbb{R})$ by Theorem 1.1, so $u_x(t, .)$ tends to $u_x(s, .)$, first in the distribution sense, then in the weak sense of probability measures.

On the other hand, given $T > s$, we now prove that $\displaystyle\sup_{0 \leq t \leq T} \int_{|x| \geq R} |x|^p \, du_x(t, x)$ goes to 0 as R goes to infinity. For this, given $\varepsilon > 0$, let R such that

$$\sup_{0 \leq h \leq T} \sup_{0 \leq t \leq T} \int_{|x| \geq R} |x|^p \, d[S_h u^0]_x(t, x) \leq \varepsilon$$

by Proposition 4.7. Let then $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $0 \leq \varphi \leq 1$ and $\varphi(x) = 0$ if $|x| \leq R$.

On one hand

$$\int_{\mathbb{R}} \varphi(x) |x|^p \, d[S_h u^0]_x(t, x) \; \to \; \int_{\mathbb{R}} \varphi(x) |x|^p \, du_x(t, x)$$

as h goes to 0 since $\varphi(x)|x|^p \in \mathcal{C}_c^\infty(\mathbb{R})$ and $[S_h u^0]_x(t, .)$ tends to $u_x(t, .)$ in distribution sense. On the other hand

$$\int_{\mathbb{R}} \varphi(x) |x|^p \, d[S_h u^0]_x(t, x) \; \leq \; \varepsilon$$

for all $0 \leq h, t \leq T$. Hence at the limit

$$\int_{\mathbb{R}} \varphi(x) |x|^p \, du_x(t, x) \; \leq \; \varepsilon$$

for all $t \leq T$, from which it follows that

$$\sup_{0 \leq t \leq T} \int_{|x| \geq R} |x|^p \, du_x(t, x) \; \leq \; \varepsilon,$$

which means that indeed $\displaystyle\sup_{0 \leq t \leq T} \int_{|x| \geq R} |x|^p \, du_x(t, x)$ goes to 0 as R goes to infinity.

From these two results we deduce the continuity result by Proposition 2.3.

5.3   Proof of Corollary 1.10

Given $t \geq 0$, $S_h u^0(t, .)$ converges to $u(t, .)$ in $L^1_{loc}(\mathbb{R})$ by Proposition 4.6, so for all $s, t, n \geq 0$ we have

$$\|u(t, .) - u(s, .)\|_{L^1([-n,n])} = \lim_{h \to 0} \|S_h u^0(t, .) - S_h u^0(s, .)\|_{L^1([-n,n])}.$$

But

$$\|S_h u^0(t, .) - S_h u^0(s, .)\|_{L^1([-n,n])} \leq \|S_h u^0(t, .) - S_h u^0(s, .)\|_{L^1(\mathbb{R})} \leq |t - s| \, \|f'\|_{L^\infty(\mathbb{R})}$$

for all $h \geq 0$ by Proposition 4.4, so letting h go to 0 we get

$$\|u(t, .) - u(s, .)\|_{L^1([-n,n])} \leq |t - s| \, \|f'\|_{L^\infty(\mathbb{R})}.$$

Since this holds for all $n \geq 0$, we obtain Corollary 1.10.

6   Extension to viscous conservation laws

In this section we let ν be a positive number and consider the viscous conservation law

$$(13) \qquad\qquad u_t + f(u)_x = \nu \, u_{xx} \qquad t > 0, \ x \in \mathbb{R}$$

with initial datum $u^0 \in L^\infty(\mathbb{R})$.

Assuming that f is a locally Lipschitz real-valued function on $\mathbb{R}$, and calling *solution* a function u in $L^\infty([0, +\infty[\times \mathbb{R})$ such that (13) holds in the sense of distributions, it is known that, given $u^0 \in L^\infty(\mathbb{R})$, there exists a unique solution u to (13). If moreover $u^0 \in \mathcal{U}$, then $u(t, .)$ also belongs to $\mathcal{U}$ for all $t \geq 0$, and the W_p contraction property stated in Theorem 1.6 in the inviscid case $\nu = 0$ still holds:

Theorem 6.1 *Given a locally Lipschitz real-valued function f on $\mathbb{R}$ and two initial data u^0 and $\tilde{u}^0$ in $\mathcal{U}$, let u and $\tilde{u}$ be the associated solutions to (13). Then, for any $t \geq 0$ and $p \geq 1$, we have (with possibly infinite values)*

$$W_p(u_x(t, .), \tilde{u}_x(t, .)) \ \leq \ W_p(u_x^0, \tilde{u}_x^0).$$

We briefly mention how this contraction property for the viscous conservation law allows to recover the same property for the inviscid equation, given in Theorem 1.6. Given some initial datum u^0 in $\mathcal{U}$ and $\nu > 0$, let indeed u_ν be the corresponding solution to the viscous equation (13). Then it is known (see [5] for instance) that $u_\nu(t, .)$ converges in $L^1_{loc}(\mathbb{R})$ to the solution $u(t, .)$ to the inviscid conservation law (1) with initial datum u^0. From this the argument already used in Section 5.1 (with $S_h u^0(t, .)$ intead of $u_\nu(t, .)$) enables to recover Theorem 1.6.

The proof of Theorem 6.1 follows the lines of Sections 4 and 5 and makes use of a time-discretization of equation (13) based on the discretization of the inviscid conservation law previously discussed. More precisely, given a time step $h > 0$, we first map $u^0 \in \mathcal{U}$ to $T_h u^0$ as in section 4.1, and then let $T_h u^0$ evolve along the heat equation on a time interval h, that is, map it to

$$\mathcal{T}_h u^0 = K_h * T_h u^0$$

where K_h is the heat kernel defined on $\mathbb{R}$ by

$$K_h(z) = \frac{1}{\sqrt{4\pi h}} \, e^{-\frac{z^2}{4h}}.$$

Then, defining an approximate solution $\mathcal{S}_h u^0$ by iterating the $\mathcal{T}_h$ operator as in Definition 4.3, we prove that Propositions 4.2 and 4.5 still hold for the new $\mathcal{T}_h$ and $\mathcal{S}_h$ operators (Note that the convolution with the heat kernel is a contraction for the Wasserstein distance of any finite order).

Proposition 4.4 only holds assuming that v is twice derivable with v'' in $L^1(\mathbb{R})$: it more precisely reads

$$\|\mathcal{S}_h v(t,.) - \mathcal{S}_h v(s,.)\|_{L^1(\mathbb{R})} \leq |t - s| \left[\|f'\|_{L^\infty(]0,1[)} + \|v''\|_{L^1(\mathbb{R})} \right].$$

As in Section 4.2, this enables to prove that, given u^0 in $\mathcal{U}$, twice derivable with $(u^0)''$ in $L^1(\mathbb{R})$, the family $(\mathcal{S}_h u^0)_h$ converges in $\mathcal{C}([0, +\infty[, L^1_{loc}(\mathbb{R}))$ to the solution of (13) with initial datum u^0.

With this convergence result in hand we follow the lines of Section 5.1 to prove Theorem 6.1 in the case of twice derivable initial data, with L^1 second derivative, while the general case follows by a density argument.

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